CS 369: Introduction to Robotics

Prof. Thao Nguyen

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HAVERFORD
COLLEGE

Admin

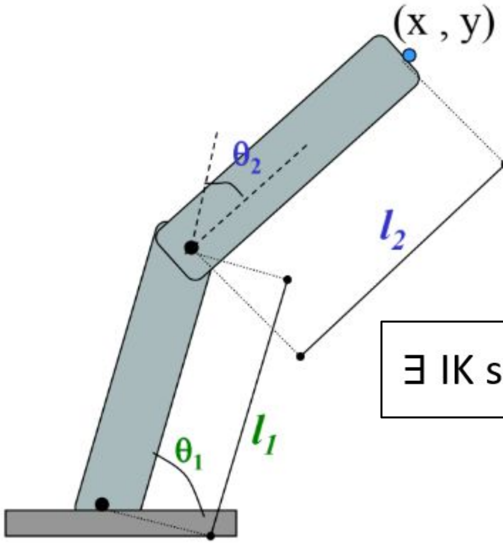
- Lab 1 due tonight

- Lab 2 posted (due next Tuesday)

Outline for today

- Inverse kinematics
  - Geometric solution
  - Algebraic solution

Inverse kinematics (IK)



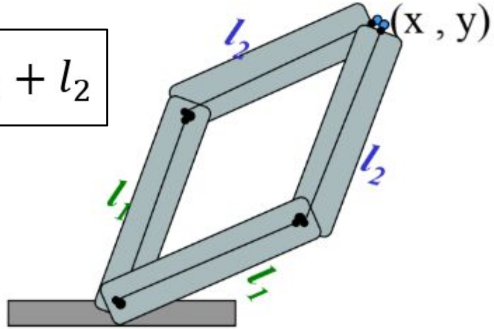
Given: l_1, l_2, x, y

Find: θ_1, θ_2

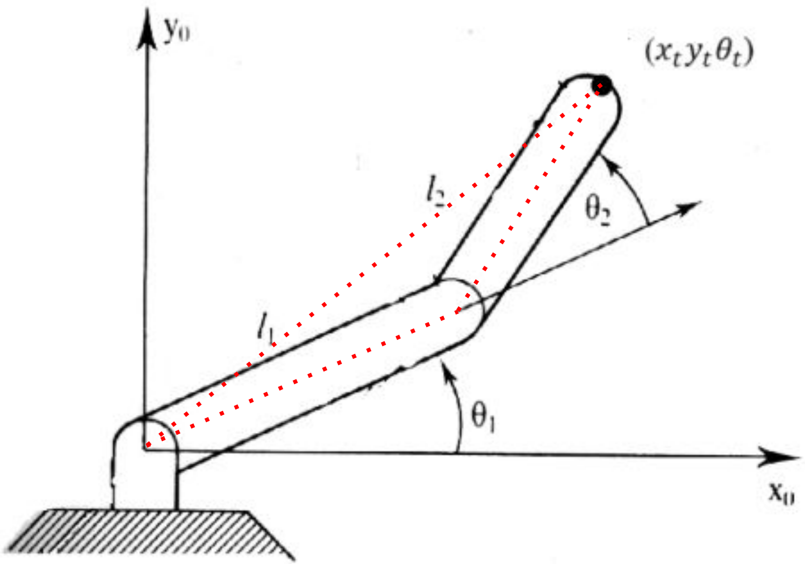
Redundancy:

A unique solution to this problem does not exist. Notice, that using the "givens" two solutions are possible. Sometimes no solution is possible.

$$\exists \text{ IK solution} \leftrightarrow l_1 - l_2 \leq \sqrt{x^2 + y^2} \leq l_1 + l_2$$



Geometric solution



Using the Law of Cosines:

$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$(x^2 + y^2) = l_1^2 + l_2^2 - 2l_1 l_2 \cos(180 - \theta_2)$$

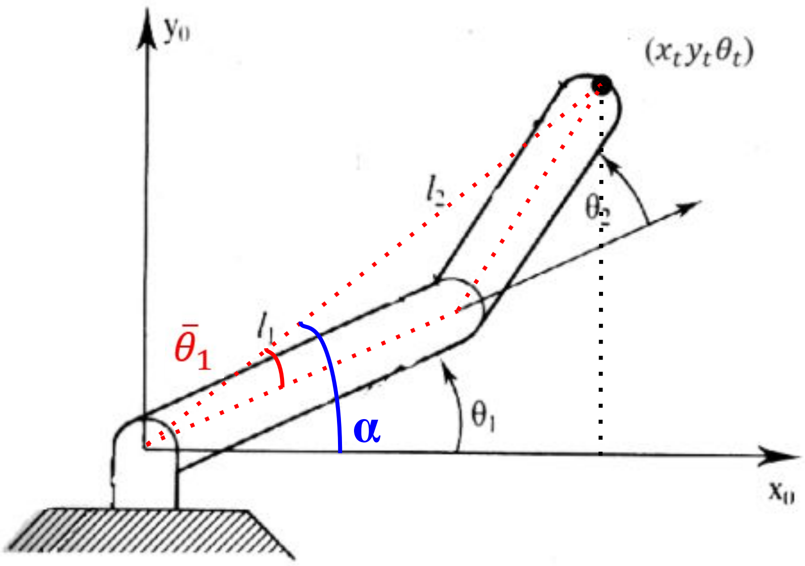
$$\cos(180 - \theta_2) = -\cos(\theta_2)$$

$$\cos(\theta_2) = \frac{x^2 + y^2 - l_1^2 - l_2^2}{2 l_1 l_2}$$

$$\theta_2 = \arccos\left(\frac{x^2 + y^2 - l_1^2 - l_2^2}{2 l_1 l_2}\right)$$

Redundant since θ_2 could be in the first or fourth quadrant.

Geometric solution



Using the Law of Sines:

$$\frac{\sin B}{b} = \frac{\sin C}{c}$$

$$\frac{\sin\bar{\theta}_1}{l_2} = \frac{\sin(180-\theta_2)}{\sqrt{x^2+y^2}} = \frac{\sin(\theta_2)}{\sqrt{x^2+y^2}}$$

$$\theta_1 = \alpha - \bar{\theta}_1$$

$$\alpha = \arctan 2\left(\frac{y}{x}\right)$$

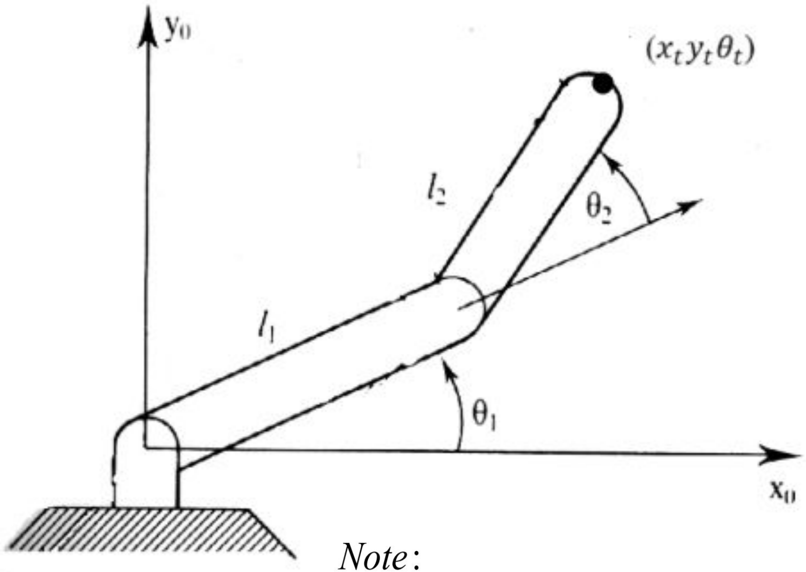
$$\theta_1 = \arctan2(y, x) - \arcsin\left(\frac{l_2 \sin(\theta_2)}{\sqrt{x^2+y^2}}\right)$$

Algebraic solution

Recall forward kinematics solution:

$$x = l_1 c_1 + l_2 c_{1+2} \qquad c_1 = \cos(\theta_1), s_1 = \sin(\theta_1)$$

$$y = l_1 s_1 + l_2 s_{1+2} \qquad c_{1+2} = \cos(\theta_1 + \theta_2)$$



$$x^2 + y^2 = \left(l_1{}^2 c_1{}^2 + l_2^2 (c_{1+2})^2 + 2 l_1 l_2 c_1 c_{1+2}\right) + \left(l_1{}^2 s_1{}^2 + l_2^2 (s_{1+2})^2 + 2 l_1 l_2 s_1 s_{1+2}\right)$$

$$= l_1{}^2 + l_2{}^2 + 2 l_1 l_2 (c_1 c_{1+2} + s_1 s_{1+2})$$

$$= l_1{}^2 + l_2{}^2 + 2 l_1 l_2 c_2$$

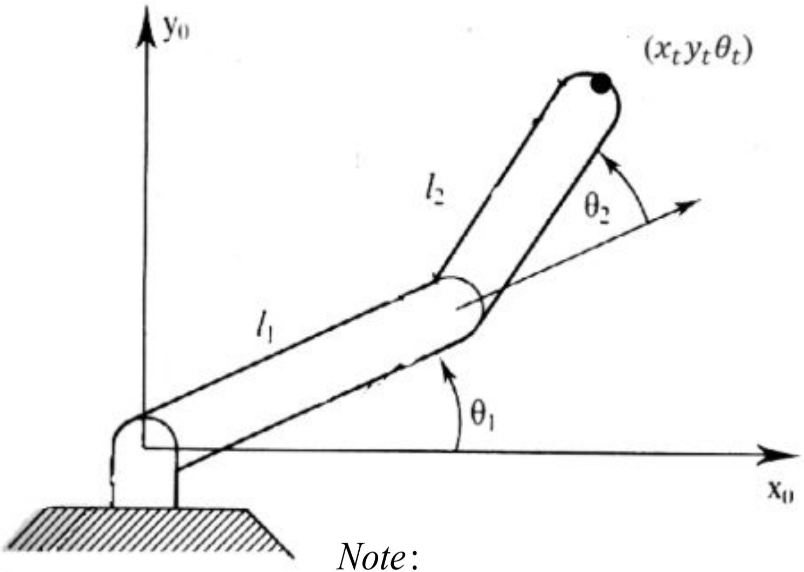
Note:

$$\cos(a \pm b) = (\cos a)(\cos b) \mp (\sin a)(\sin b)$$

$$\sin(a \pm b) = (\cos a)(\sin b) \pm (\cos b)(\sin a)$$

$$\boxed{\theta_2 = \arccos\left(\frac{x^2 + y^2 - l_1{}^2 - l_2{}^2}{2 l_1 l_2}\right)}$$

Algebraic solution



Note:

$$\cos(a \pm b) = (\cos a)(\cos b) \mp (\sin a)(\sin b)$$

$$\sin(a \pm b) = (\cos a)(\sin b) \pm (\cos b)(\sin a)$$

$$
\begin{aligned}
x &= l_1 c_1 + l_2 c_{1+2} \\
&= l_1 c_1 + l_2 c_1 c_2 - l_2 s_1 s_2 \\
&= c_1 (l_1 + l_2 c_2) - s_1 (l_2 s_2)
\end{aligned}
$$

$$c_1 = \frac{x + s_1 (l_2 s_2)}{(l_1 + l_2 c_2)}$$

$$
\begin{aligned}
y &= l_1 s_1 + l_2 s_{1+2} \\
&= l_1 s_1 + l_2 s_1 c_2 + l_2 s_2 c_1 \\
&= c_1 (l_2 s_2) + s_1 (l_1 + l_2 c_2)
\end{aligned}
$$

$$\theta_1 = \arcsin\left(\frac{y(l_1 + l_2 c_2) - x l_2 s_2}{x^2 + y^2}\right)$$

Three-link robot